

On a SUSY QM scheme for any linear homogeneous differential equation of the second order

R. KLIPPERT^{1,2} and H.C. ROSU^{1,3}

¹ International Center for Relativistic Astrophysics, Piazzale della Repubblica 10,
65100 Pescara, Italy

² Brazilian Center for Research in Physics, R. Dr. Xavier Sigaud 150 Uira, 22290-180
Rio de Janeiro RJ, Brazil

³ Instituto de Física, Univ. de Guanajuato, Apdo Postal E-143, León, Gto, Mexico

Summary. - A formal supersymmetric quantum mechanics (SUSY QM) procedure for any linear homogeneous second-order differential equation is briefly sketched up and applied to a simple exactly solvable case.

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It is quite well known how to get Riccati equations from the general form of homogeneous linear differential equation of the second order [1]

$$A(x)\frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u = 0. \quad (1)$$

It requires the transformation

$$y = \frac{1}{R(x)} \frac{1}{u} \frac{du}{dx}, \quad (2)$$

where $R(x)$ is an arbitrary function. Using $u' = Ruy$ and $u'' = Ru(y' + Ry^2 + \frac{R'}{R}y)$ in Eq. (1) leads to

$$\frac{dy}{dx} + \left(\frac{R'}{R} + \frac{B}{A} \right) y + Ry^2 = -\frac{C}{AR}. \quad (3)$$

Moreover, since $R(x)$ is an arbitrary function, one can choose it such that the coefficient of y be zero, i.e., one can end up with the simplified Riccati equation

$$\frac{dy}{dx} + R_0 y^2 = -\frac{C}{AR_0} , \quad (4)$$

where $R_0(x) = \exp[-\int^x (B/A)dx']$.

Our point here is to notice that Eq. (4) allows a direct connection with various SUSY QM schemes [2] by means of the change of independent variable $z(x) = \int^x R_0 dx'$. This leads to a Riccati equation of SUSY QM type at zero factorization energy along the curve $z(x)$

$$\frac{dy}{dz} + y^2 = V_1(z) , \quad (5)$$

where $V_1(z) = -\frac{C(z)}{A(z)R_0^2(z)}$ can be interpreted as a Schrödinger ‘potential’. The SUSY partner Riccati equation will be

$$-\frac{dy}{dz} + y^2 = V_2(z) , \quad (6)$$

where the partner ‘potential’ $V_2(z)$ is Darboux ‘isospectral’ with respect to V_1 , i.e.

$$V_2 = V_1 - 2D^2[\ln(\psi(z))] , \quad (7)$$

where $D = \frac{d}{dz}$ and ψ is a particular solution of $D^2\psi - V_1\psi = 0$.

In addition, one can think of the SUSY QM scheme based on the general Riccati solution, as first tackled by Mielnik for the harmonic oscillator case [3]. In the latter approach, one gets a one-parameter family of Darboux strictly isospectral ‘potentials’ given by

$$V_1(z; \lambda) = V_1 - 2D^2[\ln(I(z) + \lambda)] , \quad (8)$$

where $I(z) = \int^z \psi^2(t)dt$. Moreover,

$$\psi(z, \lambda) = \frac{\psi(z)}{\int^z \psi^2(t)dt + \lambda} \quad (9)$$

is the modulated Schrödinger zero mode implied by this scheme in which the Riccati integration constant λ is kept as a free parameter. In general, to get continuous solutions $\psi(z, \lambda)$, one should take care of not having a zero denominator. This leads to conditions on the possible values of λ . On the other hand, if one works with polynomial solutions there will be singularities in the log derivative of ψ . SUSY partner potentials based on

the n 'th excited state of a Schrödinger discrete spectrum problem split in $n + 1$ branches separated by the n singularities of the log derivative [4].

Even more general schemes, such as higher-order intertwining can be applied leading to a rich class of 'isospectral' solutions.

The main drawback of the formal scheme as presented in Eqs. (1)–(7) is that it is not at all easy to implement in practice because onto the z axis one does not get an easily solvable eigenvalue problem. Let us take as an example the Hermite equation

$$u'' - 2xu' + 2nu = 0 . \quad (10)$$

For this case, $A(x) = 1$, $B(x) = -2x$, and $C(x) = 2n$. Thus, $R_0(x) = e^{x^2}$ and $z(x) = \int^x e^{t^2} dt$. The potential $V_1(z) = -2n/e^{2z^2}$ does not lead to an exactly solvable problem.

We provide in the following an exactly solvable case focusing on the scheme based on the general Riccati solution. Let us take the initial equation of the form

$$D_x^2 u + (1/x)D_x u + u = 0 \quad (11)$$

Then $V_1 = -z^2$ that corresponds to an eigenvalue problem in terms of the Bessel functions, namely $\psi(z) = C_1 \sqrt{z} J_{\frac{1}{4}}(z^2/2) + C_2 \sqrt{z} Y_{\frac{1}{4}}(z^2/2)$. In Fig. 1 we present a common three-dimensional plot of the formulas (8) and (9) for the pair of superposition constants $C_1 = 0$ and $C_2 = 1$. On the other hand, in Fig. 2 we present a full axis one-dimensional plot for $\lambda = 0.2$ and $C_1 = 1$ and $C_2 = 0$.

One can also ask what is the equation in the initial x -axis corresponding to the potential $V_1(z, \lambda)$. One can easily show that such an equation has the following form

$$D_x^2 \psi(z(x), \lambda) + \frac{1}{x} D_x \psi(z(x), \lambda) - \frac{V_1(z, \lambda)}{x^2} \psi(z(x), \lambda) = 0 . \quad (12)$$

For this case, the connection between the two axes is given by $z = \ln x$.

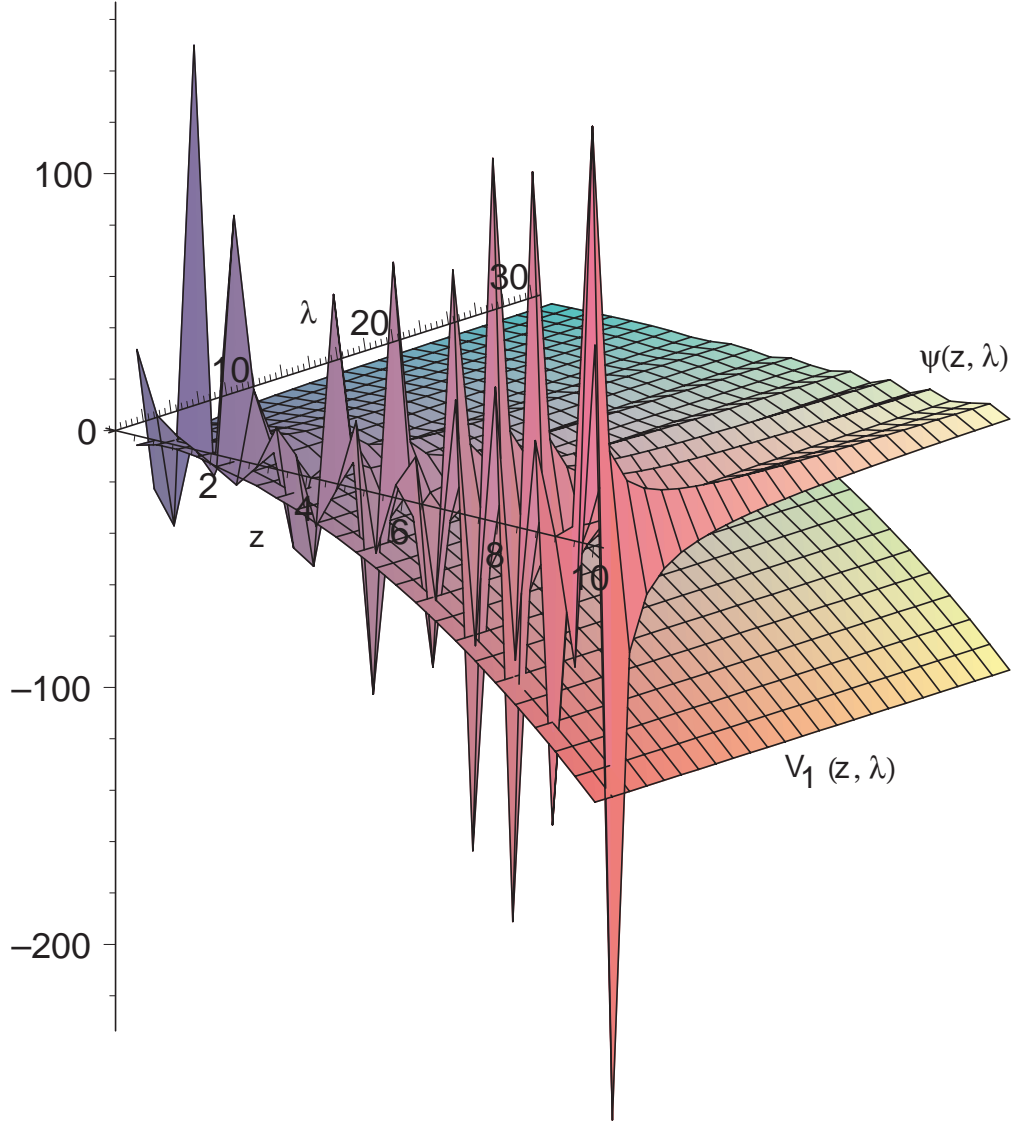


Figure 1: The potential $V_1(z, \lambda)$ and the eigenfunctions $\psi(z, \lambda)$ corresponding to $\psi(z) \sim \sqrt{z} Y_{\frac{1}{4}}(z^2/2)$ for $\lambda \in (0, 30)$. The dependence on λ is almost flat in the range of large λ .

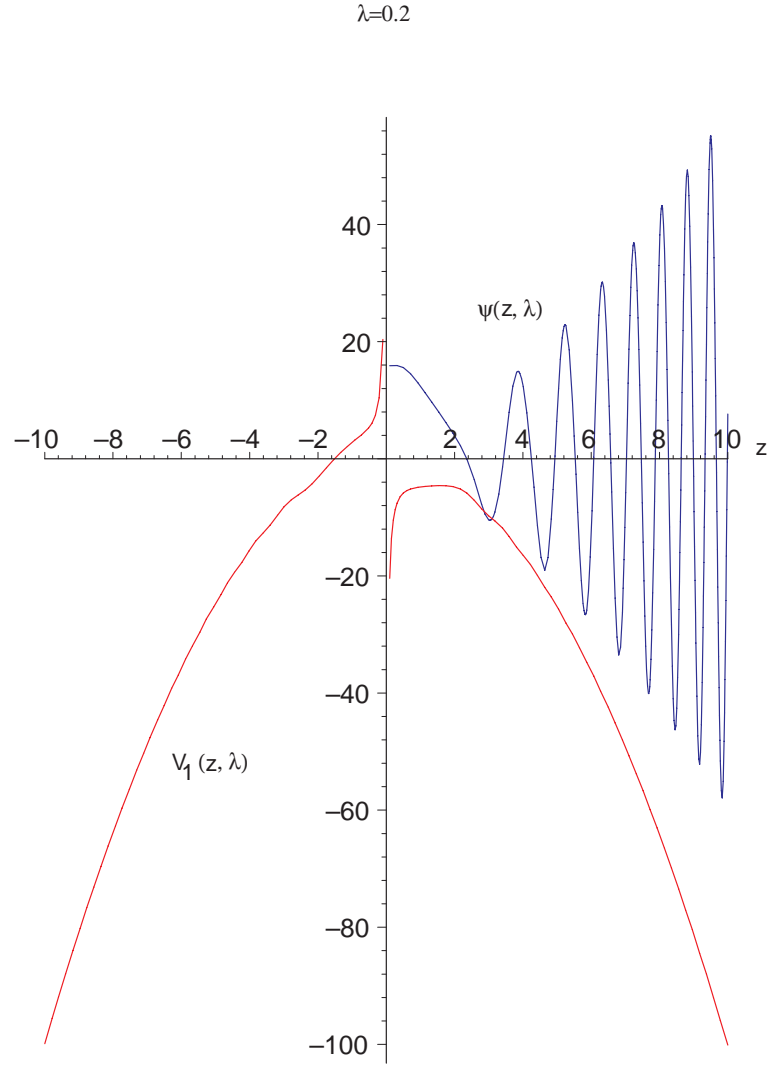


Figure 2: The eigenfunction $\psi(z, \lambda)$ corresponding to $\psi(z) \sim \sqrt{z} J_{\frac{1}{4}}(z^2/2)$ and the potential $V_1(z, \lambda)$ for $\lambda = 0.2$. The wave function seems to approach a finite value for $z = 0$, which can be explained by the presence of the opposite divergences in the potential $V_1(z, \lambda)$ at $z = 0$.

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